Etats-Unis : Uncovering and Analyzing Implicit Assumptions in Mathematics Class

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This article1 offers a brief description of a particular approach to teaching students to uncover implicit assumptions made in interpreting mathematics problems, as well as identifying and analyzing assumptions in the course of mathematical argumentation. I illustrate it with some vignettes from the mathematical practice of a group of sixth graders, and finish with some suggestions to teachers for initiating and organizing such classroom discussions.

I/ Where are implicit assumptions to be found in a mathematics class?

There is a common misconception that mathematics represents a fixed body of truths and mathematical procedures and that it always produces clear, definite, and indisputable answers. Often people use the phrase 'If you do the math, you''ll see for yourself that . . . " as if the act of "doing of math" always lays down a crystal clear path to an unambiguous defence of their claim. But mathematical problem solving, at least the kind that deals with real-world problems, relies on a process of "mathematization" or building mathematical representation/interpretation of the situation that is a formal representation, and which can be manipulated mathematically. The results of this formal manipulation are referred back to and interpreted in the context of the original problem situation,. The movement from real-world description to mathematical formal description and back requires interpretation and meaning construction of a kind that is an inherent part of all language use.

Making assumptions is a usual part of our thinking, but often those assumptions are tacit, and may go unnoticed. This makes the identification and making visible of those assumptions important when we engage in argumentation with others. It helps in judging their interpretation of a situation-that is, how it depends on assumptions made, and on unstated premises and presuppositions that inform their arguments.. Here by implicit assumptions I mean propositions whose truth is taken for granted, but is not explicitly stated. Uncovering implicit assumptions in mathematical practice is, then, the identification of such propositions.

In what follows, I will point to several processes in the course of mathematical practice in which identification of implicit assumption is essential. This list is inclusive but not limited to the processes of understanding and interpreting math problems, (and more specifically math word problems), understanding someone''s argument, evaluating an argument, or helping someone improve her argument. Learning to uncover assumptions is in fact an essential skill that is crucial in mathematical thinking, argumentation, and problem solving. Developing this skill contributes to the development of greater sophistication in the understanding, doing, and analyzing of mathematics in general.

II/ Learning to identify and evaluate assumptions in interpreting math problems

One common place to explore assumptions in a mathematics class is in problem solving, which is the backbone of mathematical practice. Some crucial aspects of problem solving are defining and interpreting problems, working with different methods to solve them, verifying solutions, and drawing conclusions. However, while students are often encouraged to work on mathematics problems "like mathematicians"-- to be persistent, investigate different approaches, and evaluate solutions--they are not typically urged to carefully and critically analyse and reformulate the problems they are given, or the ones they find in textbooks (Brown & Walter, 1983).

Mathematics word problems present situations whose descriptions often carry numerous assumptions. For example, a math problem that requires that a train''s arrival time be calculated given the start time, the distance, and the speed of the train, contains at least a tacit assumption that the train won''t break down, or make more stops than scheduled, as well as the assumption that the train''s speed is given as an average, and that the time zone remains the same for the whole trip. Students have to infer the relevant missing information in order to interpret the problem and solve it successfully. By factoring their assumptions-even the ones held unconsciously--into the given data, students are reformulating the initial problem.

The assumptions that play a part in such interpretations are often considered common sense and are rarely discussed explicitly. However, different students might rely on different assumptions in making sense of the problem situation, which would lead to different reformulations of the problem, and thus to different solutions. Therefore learning to identify and critically evaluate assumptions should be an essential part of students'' strategic competence in math problem solving-that is, the ability to formulate, represent, and solve mathematical problems (NRC, 2001). Helping students become aware of the role that assumptions play in problem solving, and learning to critically examine their own assumptions and those of their peers, is also essential to the development of dispositions of critical mathematical thinking. Generally, it is recommended that students grapple with rich, well-structured problems that allow multiple paths for problem solving, and thus enable them to explore and discuss alternative ideas (NCTM, 2000). Frederickson (1984) makes a distinction between well-structured problems that are clearly defined and can be solved by using one or multiple methods, and ill-structured problems that lack a clear formulation, and require clarification before a plan for solving them can be devised. Some elements of such problems are unspecified or ambiguous, and require that the solver reformulate the problem statement in order to develop a plan for its resolution. Often with such reformulation the interpreter, in order to make sense of the problem, herself adds information some of which is in the form of implicit assumptions. Sketchily formulated and fuzzy problems then can have pedagogical value in fostering awareness of one''s own or others'' assumptions, if such problems are presented to students with the expressed goal of examining possible interpretations and reformulations of the initial problem, and of linking those to the implicit assumptions underlying such interpretations/reformulations. Fuzzy problems can also provide a context for tracing the relations between assumptions made in the reading/interpreting a math problem and solutions considered.

III/ Identifying assumptions in mathematical argumentation

Another place where uncovering assumptions turns out to be important is the mathematical argumentation that takes place during the process of problem solving, the analysis and evaluation of solutions, or proof of mathematical statements. Often students who are not used to offering justifications for their claims omit the premises of their arguments. If the teacher presses for justification, and makes it a normative part of the classroom discourse, then students learn to regularly support their inferences with reasons. However, quite often premise-type implicit assumptions sneak in as parts of arguments. These are tacit propositions, taken for granted and used jointly with one or more other premises, as a basis for argument and action, and we will call those "gap-fillers" after Ennis (1982). Ennis (1982) offers the following example of an argument with a "gap."

If Mike is a dog, then Mike is an animal.

Therefore Mike is not a dog (p. 63).

The proposition "Mike is not an animal" could be a gap-filler here because, used jointly with the given premise it supports the conclusion. A gap is filled when one can infer without any questions a conclusion from its given premises. To fill a gap in an argument one has to identify the implicit assumption that was used in constructing the argument. Identifying such implicit assumptions-known as "used implicit assumptions" (Ennis, 1982) is essential when one attempts to understand and analyse an argument in order to figure out the position one must take towards it-to accept it, or not to accept it and offer a critique. Trying to understand a student''s used implicit assumption is not always a straightforward process, as there may be more than one alternative possible gap-fillers, which would require that the teacher probe further in order to confirm the identified used assumption instead of attributing it to the student.

Below I present excerpts from a discussion of a teacher with a group of six grade students, which shows collective mathematical argumentation with a specific focus on uncovering implicit used assumptions. The teacher ran discussions in the course of one school year that were specifically aimed at collaborative work on word problems, most of which were open-ended, that allowed for different interpretations and provoked an exploration of assumptions. Her objective was that students learn to state their mathematical ideas and offer reasons to support them. Her basic role was to orchestrate students'' contributions, and to navigate the mathematical argumentation process by asking students to clarify their statements, and to respond to and critique each other''s arguments in the interest of reaching a consensus regarding a solution of the given math problem. She also modelled a disposition for "caring for ideas" by helping students to restate and clarify theirs, and to build on each other''s arguments. She modelled and coached the practice of identifying and evaluating unstated assumptions, and the process of reaching reasonable consensus. She also helped her students to understand how assumptions inform a problem''s solution.

IV/ Focusing on assumptions: The "frog problem"

The discussion that follows revolves around the following problem: A frog finds herself at the bottom of a 30-meter deep well. Each hour she climbs 3 meters, and slips back 2 meters. How many hours would it take her to get out?

After the problem was presented to the class, suggestions immediately started to emerge, and the teacher asked the students to present and to justify their positions. Sidney thought that the frog would need 30 hours to get out of the well, since "every hour she climbs up 3 and goes down 2 m, so she climbs only one meter [per hour]." This position was challenged by Victor, who suggested that after 28 hours the frog could be out since "she won''t have more to climb." As the discussion continued, it became clear that the students had different readings of the problem situation, different interpretations of the question being asked, and were in turn proposing different solutions. The teacher asked the students to agree or disagree with the proposed interpretations and solutions, and the class split into two groups-some agreeing with Sidney and some with Victor. The reason for the different interpretations slowly emerged as well, and was articulated by Rashaad: "It doesn''t say whether the frog will decide to go back 2 m again once she gets to the top." The teacher briefly summarized Sidney''s and Victor''s positions, clarifying the underlying assumptions, and called for their evaluation." We''re making different assumptions here about what the frog is going to do. . . . Sidney is assuming that the frog will return back to the well even after she gets out. And there is Victor who thinks that once the frog is out, she won''t need to go back." This summing up of the positions and their underlying assumptions made it easier for the students to "track" and reflect on them. The students knew that they were expected to examine together all identified assumptions, and make a judgment. To that end, they spent considerable time clarifying and interpreting ambiguities--what was meant by the "frog gets out" for example, and whether the problem implies that once the frog is out it could slide back into the well, which made it unclear when exactly this "getting out" was going to occur. They were struggling to make sense of the problem situation. Some thought that the frog might get out of the well in less than 30 hours but then she "needs" to go back-assuming otherwise would violate the problem''s conditions, they thought. Others thought that the frog''s goal was to get out, and there was no reason to assume that she would slide back. The students also considered things such as whether the frog might need some sleep during this long period of climbing. After some discussion about these issues most students seemed to agree that the most plausible assumption was that the frog was climbing to get out and not to return to the well. Rush''s statement expressed well the growing consensus: "I agree with what most people are saying. Because if you''re in a well you want to climb and get out. Do you think you will jump back? You get out. Period. This is her [the frog''s] main goal. Nothing else."

But suddenly there was a new development:

Chas: But the question was, when first is she going to be out of the well? Right?

Teacher: The problem says "How many hours would it take her to get out"? How do we interpret that?

Chas: So they mean first.

Victor: I think, the question says "When it''s first going to be out." She might go back, but the question is saying first. And it''s after 28 hours.

Teacher: Chas and Victor suggest that we can clarify the question as "When is the frog first going to be out of the well"? Accept that? [Agreement among the group] Ok. Now that might help Sidney agree with the others. . . . Let''s go from here and check the calculations.

The teacher incorporated Chas''s and Viktor''s suggestion and reformulated the problem question; now it was more specific, and this clarification helped the students to see that Sidney''s assumption was now irrelevant.

While the students were grappling with clarifying the problem, the teacher encouraged the group to express their ideas, agree and disagree with their peers, and give reasons. She often identified and articulated used assumptions, for example "Nellie is assuming that it''s in the problem''s conditions that starting at the 27th m and climbing up 3 m the frog will be out of the well but then she will slide back in," and thus made them visible, and then mediated between different student''s positions, helping them reach consensus. She was active in the way of focusing students on certain issues, asking clarifying questions, summarizing, paraphrasing, and connecting ideas-for example, "Is the problem specifying whether the frog is sliding back? What is the problem not telling us? What is the assumption made by Sidney? What is a reasonable assumption to make? Are you saying that . . . . How is what Sidney saying different from what Victor is saying? Is what Rush saying in agreement with what Victor is saying?"

Once the problem question was reformulated, the discussion moved to an exploration of the candidate-solutions. Two suggestions had been given so far: 28 and 30 hours. The candidate-solution, 30 hours, was rejected as associated with Sidney''s discounted assumption that the frog was going back in the well. It appeared for a while that the only convincing answer was 28 hours. Then there was another new turn, and another plausible solution was presented, reflecting a different assumption about the frog''s mode of climbing:

Jimmy: Wait, you''re saying 28 [hours to get out], but it''s in case she climbs 3 feet at a time to reach the ground. And what if she climbs a little bit and then slides back, then again climbs a little bit, then slides down, then she wouldn''t really be out after 28 hours, would she?

Victor: The problem is saying she climbs 3 feet and then slides back 2 feet.

Bill: Well that''s the way we understood it, but it''s not quite clear.

Bud: It says "Each hour she climbs 3 feet, and slips back 2 feet."

Laura: It doesn''t say " then," so it could be Jimmy''s way too....

Teacher: So, can we assume that there are two possible ways the frog might be climbing up?

After further negotiation the students agreed that, in fact, both a strict up-three down-two and an irregular pattern amounting to the same distances were legitimate ways for the frog to climb up the well. They were then asked again by the teacher to think about possible answers for the frog''s time in each case. And as the class was approaching its end, the teacher asked:

Teacher: Ok, so we have several possible interpretations of the problem depending on what we assume about how the frog climbs. Who could summarize the conclusions that we''ve reached so far? Nora?

Nora: So, if she climbs the way we thought she did in the beginning, 28 m will be enough, but if she doesn''t first climb these 3 m and then slides back 2 m and does it differently. . . . she needs more than 28 . . . and up to 30 m.

These excerpts portray how the group dealt with understanding and solving an ambiguous problem that allowed for different interpretations, and thus solutions, due to a lack of specifics in its original formulation. Each interpretation of the original problem was aligned with an associated set of assumptions. When they were identified and clarified, the way they influenced the interpretation and the corresponding solution became apparent. The discussion "zigzagged" between negotiation of the problem formulation and reflection on and evaluation of the proposed solutions and justifications.

V/ A brief overview of another discussion: The "Clock strikes"

By way of further illustration, here is another problem that the teacher used, and a brief summary of the discussion stimulated by it that included exploration of the assumptions made, and of the ways they informed the solution: A clock strikes six times in five seconds. How long would it take the clock to strike twelve times? It was not specified in the problem whether the strikes were produced in equal time intervals or not, nor was it clear how the time was counted. Many different assumptions were broached. It was observed that different clocks might have different striking patterns. Mark remarked that we didn''t know whether the clock would continue silently after the first six strokes. Rashaad then suggested that we assume that the clock struck once every second on the second-such a clock, he thought, would strike six times in five seconds.

This was accepted by the group, and followed by a discussion about whether the clock made the first stroke and then the time count started, or whether the count started with the first stroke. Some students thought that the clock strikes first, and then the time count begins, since "the stroke says now the next second begins." Following on this assumption, Vincent offered a solution: "I think the total time [for twelve strokes] is eleven [seconds]. The stroke is per second. The clock strikes and then the time [count] starts. So, then it counts as one second when it strikes the second time, two seconds when it strikes a third time . . . and so I think it''s 11 [seconds]. "

Other students thought that the time count should start with the first stroke. "It''s a clock," Rashaad argued, "so it does everything at the same time. It''s not like the brain. . . it''s not like it''s going to think first and then do something. . . ." The teacher encouraged the students to agree or disagree with the presented positions, and the reasoning supporting them. After a thorough analysis of the reasoning involved, the group concluded that it should take the clock eleven seconds to make twelve strokes. The teacher asked them to consider whether it would matter to the final result whether the time count starts with the first stroke or after it, and they agreed that in either case it would take eleven seconds.

Over the course of the discussion, the teacher carefully led her students to explore each assumption, and the consequences of each interpretation of the problem. They did not discuss whether the time for the stroke itself should be deemed negligible or not, and if the latter, how it would affect the interpretation of the problem situation. They finished their deliberation with a keen sense of the extent to which assumptions may be buried so deep in the statement of a problem that they are not immediately visible, and that those assumptions make a distinct difference to the final solution.

I offer six more mathematics problems that would merit discussion of assumptions, and will leave them to the reader to puzzle out:

- 1) Nine dots form a square as shown below. Draw 4 straight lines through all dots without lifting your pen and without retracing any line. .
- 2) There are water lilies on the surface of a lake. Every day they double the area they cover. In 30 days they covered the whole surface of the lake. How long did it take to cover half of the area of the lake?
- 3) Max, will achieve the age of majority in 2013, and will have celebrated only 5 birthdays. In which year was he born?
- 4) A woman who had a chicken farm went to the market to sell a basket of fresh eggs. To her first customer she sold half her eggs and half an egg. To the second customer she sold half of what she had left and half an egg. And to the third customer she sold half of what she then had left and half an egg. Three eggs remained. How many did she start with?
- 5) In a certain village, 2/3 of the adult men are married to the 3/5 of the adult women. What fraction of the adults in the village are married?
- 6) Using 6 matchsticks, construct 4 congruent equilateral triangles.

VI/ Conclusion

By the end of the year-long investigative project, students were more readily able to identify and critically evaluate assumptions when they were working with word problems. They also became more aware of the fact that doing mathematics is a sense-making process; that mathematical problems are matters of interpretation and require careful examination of the data given; and that any inferences made are based upon implicit assumptions that also call for examination.

The preparation of such a project requires an appropriate selection of math problems. They have to be "noisy"-that is, contain enough ambiguity so that they allow different interpretations, which prompt different solutions. One can find such problems or devise suitable ones by removing some information from existing problems and making them ambiguous. For example, in the "frog problem," the word " then" was removed from the sentence "She climbs 3 m and then slips back 2 m" to open space for more interpretative possibilities.

A project like this also requires an environment that welcomes collaborative deliberation, open expression and respect for ideas, and a teacher''s support in articulating ideas--all of which is best achieved in a community of inquiry format, in which the teacher helps students identify and articulate assumptions in the interest of making reasoned judgments. She listens carefully, and works primarily to help students clarify, articulate, and bridge ideas, acting as a conductor rather than as a leader of the discussion. Through this form of attention to the group as a whole, the community itself assumes the role of an interlocutor, a generator of ideas, and a reflector and corrector of each student''s perspective and reasoning style. As such, collaborative argumentation, where students analyse, critique and help improve each other''s argument, is a powerful environment for exploring assumptions, and acquiring habits of mathematical thinking and problem solving.

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